CONTACT PROBLEM OF ELASTICITY THEORY FOR NARROW REGIONS

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The problem of die pressure in an elastic half-space is a classical problem of elasticity theory. Its solution is a necessary component part of designing many objects in machine and instrument building. Pressure distribution at the contact is found as a result of solving a two-dimensional first-order integral equation with a polar kernel.

In this work, an asymptotic method is suggested for narrow contact regions making it possible to reduce this equation to a set of of two unidimensional integral equations. The of first connects pressure distribution in the transverse direction with the transverse shape of elastic displacement, and the second is an equation relating to the unknown load in the section. An asymptotic approach makes it possible to substantiate the heuristic method for plane sections [1].

Analytical solutions have been obtained for two problems: contact for a die with an elliptical relationship for width; contact for a semiinfinite die with a constant load in each section. In the case of an elliptical die, with a flat base, comparison is carried out for asymptotic and precise [1] results. The method suggested rightly embraces the first term of the expansion for small parameter ε (characteristic prolation of the contact) for the precise solution of the problem.

1. The problem of pressure for a die of limited dimensions in an elastic half-space is reduced to an integral equation [1]

$$\theta\iint\limits_{G_{xy}}\frac{p'(\xi,\eta)\,d\xi\,d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}}=\frac{g'(x,y)}{L}. \tag{1.1}$$

Here $g'(x, y)$ is elastic displacement; $\theta = (1 - \nu_1^2)/(\pi E_1) + (1 - \nu_2^2)/(\pi E_2)$; ν_1, ν_2, E_1, E_2 are Poisson's ratios and elasticity moduli for the die and half-space; x and y are dimensionless Cartesian coordinates in a plane obtained from dimensional coordinates by dividing them by the L-characteristic dimension of the region G_{xy} of the contact; $p'(x, y)$ is pressure distribution.

Let the contact in the coordinate system φ, ψ be narrow. It is assumed that $x = x(\varphi, \psi)$, $y = y(\varphi, \psi)$, $(\varphi, \psi) \in G_{\varphi\psi}$ and the reflection of $G_{\varphi\psi}$ on G_{xy} prescribed by these equations is in a one-to-one way mutually without interruption differentiated with a functional determinant not equal to zero. Then C exists such that $|x_\varphi|, |y_\varphi|, |x_\psi|, |y_\psi| \leqslant C$. In addition, it is assumed that the following inequalities are fulfilled:

$$|x_\varphi(\varphi_1,\psi)-x_\varphi(\varphi_2,\psi)|\leqslant C_1|\varphi_1-\varphi_2|^{\gamma_1},\quad |x_\varphi(\varphi,\psi_1)-x_\varphi(\varphi,\psi_2)|\leqslant C_1|\psi_1-\psi_2|^{\gamma_2}, \tag{1.2}$$

$$|y_\varphi(\varphi_1,\psi)-y_\varphi(\varphi_2,\psi)|\leqslant C_1|\varphi_1-\varphi_2|^{\gamma_3},\quad |y_\varphi(\varphi,\psi_1)-y_\varphi(\varphi,\psi_2)|\leqslant C_1|\psi_1-\psi_2|^{\gamma_4},\quad \gamma_1,\gamma_2,\gamma_3,\gamma_4>0.$$

Let, in variables of φ and ψ, region $G_{\varphi\psi}$ depend on ε so that in variables $u = \varphi$, $v = (\psi - V(\varphi))/\varepsilon$, region G_{uv} corresponding to $G_{\varphi\psi}$, has the form $G_{uv} = \{(u, v)\colon u^- \leqslant u \leqslant u^+, v^-(u) \leqslant v \leqslant v^+(u)\}$ ($V(\varphi)$, $v^+(u)$ are continuous functions). The limit of region $G_{\varphi\psi}$ with $\varepsilon \to 0$ is a line $\psi = V(\varphi)$ which in future we shall call the skeletal line.

In variables u and v Eq. (1.1) is written in the form

$$\iint\limits_{G_{uv}}\frac{p(\xi,\eta)\,d\xi\,d\eta}{|\mathbf{r}(u,v)-\mathbf{r}(\xi,\eta)|}=\frac{g(u,v)}{\varepsilon}, \tag{1.3}$$

where $\mathbf{r} = (x(u, v), y(u, v))$; $g = g'/L$; $p = \theta p'\partial(x, y)/\partial(\varphi, \psi)$. It is noted that in the left-hand part of (1.3), just as in the right-hand part, there is, small parameter ε since $|x_v|, |y_v| \leqslant C\varepsilon$.

We introduce the notations $a = \max\limits_u |v^+(u) - v^-(u)|$, $\mathbf{R}(u) = \mathbf{r}(u, 0)$ ($\mathbf{R}(u)$ governs the hodograph of the skeletal line). Let $\mathbf{r} = \mathbf{R}(u)$ be a smooth curve. Then as follows from the Lagrangian

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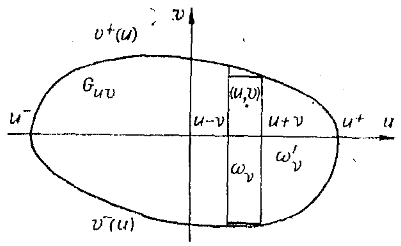

Fig. 1

finite increment equation there exists $M > 0$: $|\mathbf{R}(u+\delta) - \mathbf{R}(u)| \leqslant M|\delta|$. It is also assumed that $m > 0$, $\delta^* > 0$ exist such that $|\mathbf{R}(u+\delta) - \mathbf{R}(u)| \geqslant m|\delta|$ for any $|\delta| < \delta^*$.

2. We obtain an asymptotic expression of the integral in (1.3) with $\varepsilon \to 0$ assuming that $p = O(1)$ is a continuous function in G_{uv}. We fix point $(u, v) \in G_{uv}$, $u \neq u^\pm$ (see Fig. 1). We select infinitely small ν so that $u - \nu > u^-$, $u + \nu < u^+$, $\nu < \delta^*$, $\varepsilon = o(\nu)$, $m\nu > 4\sqrt{2}Ca\varepsilon$. We present the integral in the form of the sum of integrals for the regions $\omega_\nu = G_{uv} \cap \{(u', v)$: $|u' - u| < \nu\}$ (I_1) and $\omega_\nu' = G_{uv} \backslash \omega_\nu$ (I_2), and in each term we separate the principal part. The following equation is correct:

$$I_2 = \int\limits_{u^-}^{u-\nu} d\xi \int\limits_{v^-(\xi)}^{v^+(\xi)} \frac{p(\xi, \eta)\, d\eta}{|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)|} + \int\limits_{u+\nu}^{u^+} d\xi \int\limits_{v^-(\xi)}^{v^+(\xi)} \frac{p(\xi, \eta)\, d\eta}{|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)|}. \tag{2.1}$$

We transform the denominator of the integrands:

$$|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)| = |\mathbf{R}(u) - \mathbf{R}(\xi) + \mathbf{A} + \mathbf{B}|, \quad \mathbf{A} = \mathbf{r}(u, v) - \mathbf{r}(u, 0),\ \mathbf{B} = \mathbf{r}(\xi, 0) - \mathbf{r}(\xi, \eta).$$

According to the Lagrangian theorem, the inequality $|\mathbf{A}|, |\mathbf{B}| \leqslant \sqrt{2}Ca\varepsilon$, is correct, by the use of which we obtain

$$\Delta = \left|\frac{1}{|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)|} - \frac{1}{|\mathbf{R}(u) - \mathbf{R}(\xi)|}\right| = [|\mathbf{A} + \mathbf{B}|^2 + 2((\mathbf{A} + \mathbf{B}),$$
$$(\mathbf{R}(u) - \mathbf{R}(\xi)))]/|\mathbf{R}(u) - \mathbf{R}(\xi) + \mathbf{A} + \mathbf{B}|/|\mathbf{R}(u) - \mathbf{R}(\xi)|/(|\mathbf{R}(u) - \mathbf{R}(\xi)| +$$
$$+ |\mathbf{R}(u) - \mathbf{R}(\xi) + \mathbf{A} + \mathbf{B}|) \leqslant 16\frac{2C^2a^2\varepsilon^2 + \sqrt{2}\, CMa(u^+ - u^-)\varepsilon}{3m^3\nu^3} \equiv \lambda'(\varepsilon, \nu).$$

Here the inequality $|\mathbf{R}(u) - \mathbf{R}(\xi) + \mathbf{A} + \mathbf{B}| \geqslant |\mathbf{R}(u) - \mathbf{R}(\xi)| - |\mathbf{A}| - |\mathbf{B}| \geqslant m\nu/2$ is used.

We select ν, which until now has remained quite arbitrary, so that $\varepsilon = o(\nu^3)$. Then $\Delta \to 0$ with $\varepsilon \to 0$ uniformly for $(\xi, \eta) \in \omega_\nu'$. We designate $q(u) = \int\limits_{v^-(u)}^{v^+(u)} p(u, \eta)\, d\eta$. In view of the continuity of p there exists $M_q > 0$ with $|q| < M_q$. Then principal part I_2 is given by the expression

$$I_2 \sim \int\limits_{u^-}^{u-\nu} \frac{q(\xi)\, d\xi}{|\mathbf{R}(\xi) - \mathbf{R}(u)|} + \int\limits_{u+\nu}^{u^+} \frac{q(\xi)\, d\xi}{|\mathbf{R}(\xi) - \mathbf{R}(u)|}. \tag{2.2}$$

In fact, since $\Delta \to 0$, the difference modulus for the right-hand parts of (2.1) and (2.2) does not exceed $(u^+ - u^-)M_q\lambda'(\varepsilon, \nu) \to 0$.

We move to estimating integral I_1:

$$I_1 = \iint\limits_{\omega_\nu} \frac{p(u, \eta)\, d\xi\, d\eta}{|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)|} + \iint\limits_{\omega_\nu} \frac{p(\xi, \eta) - p(u, \eta)}{|\mathbf{r}(u, v) - \mathbf{r}(\xi, \eta)|} d\xi\, d\eta. \tag{2.3}$$

Since $p = O(1)$, $|p(\xi, \eta) - p(u, \eta)| \leqslant w(\nu, p, G_{uv})$ (w is the continuity modulus for p at contact G_{uv}) and according to the Cantor theorem $w \to 0$ with $\varepsilon \to 0$, then the second integral in (2.3) may be ignored in comparison with the first.

In region ω_ν we write the maximum rectangle $Q_\nu = \{(u', v)$: $|u' - u| \leqslant \nu$, $v_1^-(u, \nu) \leqslant v \leqslant v_1^+(u, \nu)\}$, where $v_1^+(u, \nu) = \min v^+(u)$, $v_1^-(u, \nu) = \max v^-(u)$, $u \in [u - \nu, u + \nu]$. Due to continuity of the $v^\pm(u)$ the values of $|v_1^+(u, \nu) - v^+(u)|$, $|v_1^-(u, \nu) - v^-(u)|$ are asymptotically small. Then meas $(\omega_\nu \setminus Q_\nu)$ is asmymptotically small compared with meas (Q_ν), and this means that

$$\iint\limits_{\omega_\nu \setminus Q_\nu} \frac{p(u,\eta)\,d\xi\,d\eta}{|\mathbf{r}(u,v)-\mathbf{r}(\xi,\eta)|} = o\left(\iint\limits_{Q_\nu} \frac{p(u,\eta)\,d\xi\,d\eta}{|\mathbf{r}(u,v)-\mathbf{r}(\xi,\eta)|}\right). \tag{2.4}$$

We transform the denominator in the right-hand part of (2.4) for $(\xi,\eta)\in Q_\nu$. By using a finite increment equation we obtain

$$\mathbf{r}(\xi,\eta) = \mathbf{r}(u,\eta) + \left[\mathbf{R}'(u) + \begin{pmatrix}\Delta_x\\ \Delta_y\end{pmatrix} + \begin{pmatrix}\Delta_x'\\ \Delta_y'\end{pmatrix}\right](\xi-u), \tag{2.5}$$

where $\Delta_x = x_u(\xi',\eta) - x_u(u,\eta)$; $\Delta_x' = x_u(u,\eta) - x_u(u,0)$; $\Delta_y = y_u(\xi'',\eta) - y_u(u,\eta)$; $\Delta_y' = y_u(u,\eta) - y_u(u,0)$; ξ', ξ'' are points lying between ξ and u. As follows from (1.2), the following inequalities are correct: $|\Delta_x| \leqslant C_1\nu^{\gamma_1}$, $|\Delta_x'| \leqslant C_1 a^{\gamma_2}\varepsilon^{\gamma_2}$, $|\Delta_y| \leqslant C_1\nu^{\gamma_3}$, $|\Delta_y'| \leqslant C_1 a^{\gamma_4}\varepsilon^{\gamma_4}$. Considering (2.5) we find that $\mathbf{r}(\xi,\eta) = \mathbf{r}(u,\eta) + (\mathbf{R}'(u)+\mathbf{b})(\xi-u)$, $|\mathbf{b}| \leqslant C_1(\nu^{\gamma_1} + a^{\gamma_2}\varepsilon^{\gamma_2} + \nu^{\gamma_3} + a^{\gamma_4}\varepsilon^{\gamma_4}) \to 0$.

We present the integral with respect to Q_ν in the form

$$\iint\limits_{Q_\nu} \frac{p(u,\eta)\,d\xi\,d\eta}{|\mathbf{r}(u,v)-\mathbf{r}(\xi,\eta)|} = \iint\limits_{Q_\nu} p(u,\eta)\left[\,|\mathbf{r}(u,v)-\mathbf{r}(u,\eta)-(\xi-u)(\mathbf{R}'(u)+\mathbf{b})|^{-1} - |\mathbf{r}(u,v)-\mathbf{r}(u,\eta)-\mathbf{R}'(u)(\xi-u)|^{-1}\right]d\xi\,d\eta + \iint\limits_{Q_\nu} \frac{p(u,\eta)\,d\xi\,d\eta}{|\mathbf{r}(u,v)-\mathbf{r}(u,\eta)-\mathbf{R}'(u)(\xi-u)|}. \tag{2.6}$$

In the right-hand part of (2.6) the first integral is asymptotically small in comparison with the second, in view of the uniform smallness of $|\mathbf{b}|$. Then, considering that $|\mathbf{r}(u,v)-\mathbf{r}(u,\eta)|$ is a value of the order of ε (since $\mathbf{r}_v \sim \varepsilon$), by introducing the notation

$$s = \frac{((\mathbf{r}(u,v)-\mathbf{r}(u,\eta)),\ \mathbf{R}'(u))}{\varepsilon\,|\mathbf{R}'(u)|^2}, \quad t^2 = \frac{|\mathbf{r}(u,v)-\mathbf{r}(u,\eta)|^2}{\varepsilon^2\,|\mathbf{R}'(u)|^2} \tag{2.7}$$

(s and t are of the order of unity) and by calculating the second integral in the right-hand part of (2.6), we have

$$I_1 \sim \frac{1}{|\mathbf{R}'(u)|}\int\limits_{v_1^-(u,\nu)}^{v_1^+(u,\nu)} p(u,\eta)\,\operatorname{arsh}\frac{\xi-\varepsilon s}{\varepsilon\sqrt{t^2-s^2}}\Bigg|_{\xi=-\nu}^{\xi=\nu} d\eta. \tag{2.8}$$

Due to the fact that $\varepsilon = O(\nu)$, values of $(\pm\nu-\varepsilon s)/(\varepsilon\sqrt{t^2-s^2})$ are asymptotically large. Then by using the relationship arsh $x = \operatorname{sgn} x \ln(2|x|) + O(x^{-2})$, $x\to\infty$, we find that the principal part of the expression in square brackets of (2.8) equals $2\ln[2\nu/(\varepsilon\sqrt{t^2-s^2})]$. By using the asymptotic smallness of values $|v_1^\pm - v^\pm|$ and substituting the upper and lower integration limits by v^+ and v^-, we find that

$$I_1 \sim \frac{2}{|\mathbf{R}'(u)|}\int\limits_{v^-(u)}^{v^+(u)} p(u,\eta)\ln\frac{2\nu}{\varepsilon\sqrt{t^2-s^2}}\,d\eta. \tag{2.9}$$

We transform the right-hand part of expression (2.2):

$$I_2 \sim \int\limits_{u^-}^{u-\nu}\frac{q(\xi)-q(u)}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}d\xi + \int\limits_{u+\nu}^{u^+}\frac{q(\xi)-q(u)}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}d\xi + q(u)\int\limits_{u^-}^{u-\nu}\frac{d\xi}{|\mathbf{R}(\xi)-\mathbf{R}(u)|} + q(u)\int\limits_{u+\nu}^{u^+}\frac{d\xi}{|\mathbf{R}(\xi)-\mathbf{R}(u)|} - \frac{q(u)}{|\mathbf{R}'(u)|}\left\{\int\limits_{u^-}^{u-\nu}\frac{d\xi}{|\xi-u|} + \int\limits_{u+\nu}^{u^+}\frac{d\xi}{|\xi-u|}\right\} + \frac{q(u)}{|\mathbf{R}'(u)|}\ln(u^+-u)(u-u^-) - \frac{2q(u)}{|\mathbf{R}'(u)|}\ln\nu. \tag{2.10}$$

The first two integrals with $\varepsilon\to 0$ have the limit $\int\limits_{u^-}^{u^+}\frac{q(\xi)-q(u)}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}d\xi$, and the limit of the four subsequent integrals is $q(u)\,\mathrm{v.p.}\int\limits_{u^-}^{u^+}\left\{\frac{1}{|\mathbf{R}(\xi)-\mathbf{R}(u)|} - \frac{1}{|\mathbf{R}'(u)(\xi-u)|}\right\}d\xi$. Taking this into account and also (2.9), it is possible to write Eq. (1.3) in the form

$$\frac{2}{|\mathbf{R}'(u)|}\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln\frac{2}{\sqrt{t^2-s^2}}\,d\eta + \text{v.p.}\int_{u^-}^{u^+}\frac{q(\xi)-q(u)}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}\,d\xi +$$
$$+ q(u)\,\text{v.p.}\int_{u^-}^{u^+}\left\{\frac{1}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}-\frac{1}{|\mathbf{R}'(u)(\xi-u)|}\right\}d\xi + \frac{q(u)}{|\mathbf{R}'(u)|}\ln\frac{(u^+-u)(u-u^-)}{\varepsilon^2} = \frac{g(u,v)}{\varepsilon}. \tag{2.11}$$

It is noted that in the final equation there is no ν (which during derivation is selected with a reasonable degree of arbitrariness).

3. We write Eq. (2.11) for two coordinate systems.

A. An affine coordinate system φ, ψ. Let $(\mathbf{e}_\varphi, \mathbf{e}_\psi)$, $(\mathbf{e}_x, \mathbf{e}_y)$ bases, such that the vectors have unit length, and the following equations hold: $\mathbf{e}_\varphi = \mathbf{e}_x$, $\mathbf{e}_\psi = \mathbf{e}_x\cos\alpha + \mathbf{e}_y\sin\alpha$; where α is the angle between $\mathbf{e}_\psi$ and $\mathbf{e}_x$. Then $\mathbf{r} = \varphi e_\varphi + \psi e_\psi$, $\varphi = x - y\cot\alpha$, $\psi = y/\sin\alpha$. Let also $V(\varphi) \equiv 0$ (the skeletal line is a section lying on straight line $\psi = 0$), and consequently, $\mathbf{R}(\varphi) = e_\varphi\varphi$. In this way $p = \theta\sin\alpha\, p'$. Since $v = \psi/\varepsilon$, $u = \varphi$, then $\mathbf{r}(u, v) - \mathbf{r}(u, \eta) = \varepsilon(v - \eta)\mathbf{e}_\psi$, $\mathbf{R}'(u) = \mathbf{e}_\varphi$, $s = (v-\eta)\cos\alpha$, $t^2 = (v-\eta)^2$

Equation (2.11) takes the form

$$\text{v.p.}\int_{u^-}^{u^+}\frac{q(\xi)-q(u)}{|\xi-u|}\,d\xi + q(u)\ln\frac{4(u^+-u)(u-u^-)}{\sin^2\alpha\varepsilon^2} = 2\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln|v-\eta|\,d\eta + \frac{g(u,v)}{\varepsilon}. \tag{3.1}$$

B. A polar coordinate system $x = \psi\cos\varphi$, $y = \psi\sin\varphi$. Let the skeletal line lie on a circle of unit radius. Then $u = \varphi$, $v = (\psi - 1)/\varepsilon$, $\mathbf{R}(u) = \mathbf{e}_x\cos u + \mathbf{e}_y\sin u$, $\mathbf{R}'(u) = -\mathbf{e}_x\sin u + \mathbf{e}_y\cos u$, $\mathbf{r}(u, v) = (1+\varepsilon v)\mathbf{R}(u)$, $\mathbf{r}(u, v) - \mathbf{r}(u, \eta) = \varepsilon(v-\eta)\mathbf{R}(u)$, $\mathbf{R}(\xi) - \mathbf{R}(u) = (\cos\xi - \cos u)\mathbf{e}_x + (\sin\xi - \sin u)\mathbf{e}_y$, $s = 0$, $t^2 = (v-\eta)^2$, $\dfrac{1}{|\mathbf{R}(\xi)-\mathbf{R}(u)|} - \dfrac{1}{|\mathbf{R}'(u)(\xi-u)|} = \dfrac{1}{2\left|\sin\frac{\xi-u}{2}\right|} - \dfrac{1}{|\xi-u|}$

Equation (2.11) is transformed to

$$\text{v.p.}\int_{u^-}^{u^+}\frac{q(\xi)-q(u)}{2\left|\sin\frac{\xi-u}{2}\right|}\,d\xi + q(u)\ln\frac{64\left|\operatorname{tg}\frac{u^+-u}{4}\operatorname{tg}\frac{u-u^-}{4}\right|}{\varepsilon^2} = 2\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln|v-\eta|\,d\eta + \frac{g(u,v)}{\varepsilon}. \tag{3.2}$$

We note one property of Eqs. (3.1) and (3.2). It assumed that a solution for $p(u, v)$ is found with prescribed $g(u, v)$, $v^\pm(u)$. Then for any continuous function $z(u)$ $p_1(u, v) = p(u, v - z(u))$ there will be a solution of the problem with $g_1(u, v) = g(u, v - z(u))$, $v_1^\pm(u) = v^\pm(u) - z(u)$, i.e., distortion of the contact region with retention of the width in each section, u = const leads to the same distortion for pressure distribution.

4. We study Eq. (2.11). If we subtract from (2.11) a similar equation written with $v = v^0(u) = (v^+(u) + v^-(u))/2$, then we obtain an equation of plane elasticity theory:

$$\frac{2}{|\mathbf{R}'(u)|}\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln\sqrt{\frac{t^2(u, v^0(u), \eta) - s^2(u, v^0(u), \eta)}{t^2(u, v, \eta) - s^2(u, v, \eta)}}\,d\eta = \frac{1}{3}[g(u,v) - g(u, v^0(u))], \tag{4.1}$$

which connects pressure distribution in each section u = const with the shape of elastic displacements in this section. As a result of solving (4.1), $p(u, v)$ is expressed in terms of $v^-(u)$, $v^+(u)$, $q(u)$. By placing it in (2.11) with $v = v^0(u)$ we have a unidimensional equation with respect to $q(u)$:

$$\text{v.p.}\int_{u^-}^{u^+}\frac{q(\xi)-q(u)}{|\mathbf{R}(\xi)-\mathbf{R}(u)|}\,d\xi + q(u)\int_{u^-}^{u^+} d\xi\left\{\frac{1}{|\mathbf{R}(\xi)-\mathbf{R}(u)|} - \frac{1}{|\mathbf{R}'(u)(\xi-u)|}\right\} + \frac{q(u)}{|\mathbf{R}'(u)|}\ln\frac{4(u^+-u)(u-u^-)}{\varepsilon^2} =$$
$$= \frac{g(u, v^0(u))}{\varepsilon} + \frac{2}{|\mathbf{R}'(u)|}\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln\sqrt{t^2(u, v^0(u), \eta) - s^2(u, v^0(u), \eta)}\,d\eta. \tag{4.2}$$

Thus, the original two-dimensional integral equation breaks down into two unidimensional equations ((4.1) and (4.2)) solved successively.

We consider the solution of (3.1). For an affine coordinate system, (4.2) is written in the form of the equation

$$\int_{v^-(u)}^{v^+(u)} p(u,\eta)\ln\left|\frac{v^0(u)-\eta}{v-\eta}\right| d\eta = \frac{1}{2\varepsilon}[g(u,v)-g(u,v^0(u))],$$

whose solution is [2]

$$p(u,v) = -\frac{1}{2\pi^2\varepsilon\sqrt{(v^+-v)(v-v^-)}}\int_{v^-}^{v^+}\frac{\sqrt{(v^+-\eta)(\eta-v^-)}}{\eta-v}\times g_v(u,\eta)\,d\eta + \frac{q}{\pi\sqrt{(v^+-v)(v-v^-)}}\quad (g_v(u,v) = \partial g(u,v)/\partial v). \tag{4.3}$$

We place (4.3) in (3.1) assuming that $v = v^0(u)$, and we use a lemma [2] about substitution of the order of integration in the integral in the right-hand part of Equality (3.1). By using a tabulated integral [3] $\int_{-c}^{c}\frac{\ln|x|}{\sqrt{c^2-x^2}}dx = \pi\ln\frac{c}{2}$ we find the contribution of the second term from (4.3) in the right-hand part of (3.1):

$$\int_{v^-}^{v^+}\frac{q\ln|v^0-\eta|}{\pi\sqrt{(v^+-\eta)(\eta-v^-)}}d\eta = q\ln\frac{(v^+-v^-)^2}{16}. \tag{4.4}$$

The contribution of the first term from (4.3) is

$$\int_{v^-}^{v^+} g_v(u,\eta)\sqrt{(v^+-\eta)(\eta-v^-)}\left[\frac{1}{\pi^2\varepsilon}\int_{v^-}^{v^+}\frac{\ln|v-v^0|\,dv}{(v-\eta)\sqrt{(v^+-v)(v-v^-)}}\right]d\eta.$$

The expression in square brackets is found by means of residue theory and tabulated integrals [3]:

$$\frac{1}{\pi^2\varepsilon}\int_{v^-}^{v^+}\frac{\ln|v-v^0|\,dv}{(v-\eta)\sqrt{(v^+-v)(v-v^-)}} = \frac{\operatorname{sgn}(\eta-v^0)}{\varepsilon\sqrt{(v^+-v)(v-v^-)}}\left[1-\frac{1}{\pi}\arccos\left(-2\frac{|\eta-v^0|}{(v^+-v^-)}\right)\right]. \tag{4.5}$$

Taking account of (4.4) and (4.5) the integral in the right-hand part of (3.1) is

$$\frac{1}{\varepsilon}\int_{v^-}^{v^+} g_v(u,\eta)\operatorname{sgn}(\eta-v^0)\,d\eta - \frac{1}{\pi\varepsilon}\int_{v^-}^{v^+} g_v(u,\eta)\times\operatorname{sgn}(\eta-v^0)\arccos\left(-2\frac{|\eta-v^0|}{(v^+-v^-)}\right)d\eta. \tag{4.6}$$

The first term in (4.6) equals $[g(u, v^+(u)) + g(u, v^-(u)) - 2g(u, v^0(u))]/\varepsilon$. The second in (4.6) is in the form of the sum of two integrals with respect to sections $[v^-, v^0]$, $[v^0, v^+]$, which are taken by parts. As a result of this

$$\frac{1}{\varepsilon}\left[g(u,v^0(u)) - g(u,v^+(u)) - g(u,v^-(u)) + \frac{1}{\pi}\int_{v^-}^{v^+}\frac{g(u,v)\,dv}{\sqrt{(v^+-v)(v-v^-)}}\right].$$

Equation (3.1) is written in final form

$$\text{v.p.}\int_{u^-}^{u^+}\frac{q(\xi)-q(u)}{|\xi-u|}d\xi + q(u)\ln\left[\frac{64(u^+-u)(u-u^-)}{\sin^2\alpha\,(v^+(u)-v^-(u))^2\varepsilon^2}\right] = \frac{1}{\pi\varepsilon}\int_{v^-(u)}^{v^+(u)}\frac{g(u,v)\,dv}{\sqrt{(v^+(u)-v)(v-v^-(u))}}.$$

Without any loss of generality it is assumed that $u^\pm = \pm 1$. Let the displacement g be independent of v and $v^+(u) - v^-(u) = 2\sqrt{1-u^2}$ (particularly in considering a die of elliptical shape). With these assumptions $p(u,v) = \frac{q(u)}{\pi\sqrt{(v^+(u)-v)(v-v^-(u))}}$ and Eq. (4.7) takes a simpler form:

$$\int_{-1}^{1}\frac{q(\xi)-q(u)}{|\xi-u|}d\xi + q(u)\ln\frac{16}{\varepsilon^2} = \frac{g(u)}{\varepsilon}. \tag{4.8}$$

This equation has a polynomial solution. We assume that $g(u) = b_n u + b_{n-1}u^{n-1} + \ldots + b_0$ and we shall find the solution of $q(u) = a_n u^n + a_{n-1}u^{n-1} + \ldots + a_0$. We use the relationship

$$\int_{-1}^{1} \frac{\xi^n - u^n}{|\xi - u|} d\xi = -2\left(\sum_{k=1}^{n} \frac{1}{k}\right) u^n + \sum_{k=0}^{n-1} \frac{u^k}{n-k}\left[(-1)^{n-k} + 1\right].$$

Then in order to determine a_j a set of linear equations is obtained

$$a_k\left(\ln\frac{16}{\varepsilon^2} - 2\sum_{j=1}^{k}\frac{1}{j}\right) + \sum_{j=k+1}^{n} \frac{(-1)^{j-k}+1}{j-k} a_j = \frac{b_k}{\varepsilon}$$

with an upper triangular matrix. In the particular case $g(u) = \text{const}$, $q(u) = g/(\varepsilon \ln(16/\varepsilon^2))$.

For an elliptical die with a flat base, there is an equation [1] $q = \dfrac{g}{2\varepsilon F\left(\frac{\pi}{2}, \sqrt{1-\varepsilon^2}\right)}$ (F is a first-order complete elliptical integral). As follows from [4], with $\varepsilon \to 0$, $F \sim \varepsilon \ln(4/\varepsilon)$. Thus the accurate and approximate solutions are asymptotically close.

We study the case of unbounded regions. Let $G_{uv} = \{(u, v): u \geqslant 0, |v - v^0(u)| \leqslant d(u)\}$. We shall consider the equation

$$\lim_{T\to\infty} \iint_{G_T} p(\xi, \eta)\left[\frac{1}{\sqrt{(u-\xi)^2 + \varepsilon^2 (v-\eta)^2}} - \frac{1}{\sqrt{(1-\xi)^2 + \varepsilon^2 (v^0(1) - \eta)^2}}\right] d\xi\, d\eta = \frac{g(u)}{\varepsilon}, \tag{4.9}$$

where $G_T = G_{uv} \cap \{(u, v): u \leqslant T\}$; $v = y/\varepsilon$; $u = x$ (x, y are rectangular coordinates). Without considering the second term in square brackets under the integral, the limit in the left-hand part may be considered infinite. Presence of this term leads to subtraction of an "infinite" displacement at fixed point $(u, v) = (1, v^0(1))$.

The expression for the integral in the left-hand part of (4.9) for fixed T is obtained from (4.7) with $u^- = 0$, $u^+ = T$. Then (4.9) is written in the form

$$\lim_{T\to\infty}\left\{\int_0^T \frac{q(\xi) - q(u)}{|\xi - u|} d\xi - \int_0^T \frac{q(\xi) - q(1)}{|\xi - 1|} d\xi + q(u)\ln\frac{u(T-u)}{\varepsilon^2} - q(1)\ln\frac{T-1}{\varepsilon^2}\right\} = 2q(u)\ln\frac{d(u)}{4} - 2q(1)\ln\frac{d(1)}{4} + \frac{g(u)}{\varepsilon}.$$

We shall take a solution of (4.10) with $q(u) = \text{const}$. Then $\lim\limits_{T\to\infty}\left\{\ln u + \ln\frac{T-u}{T-1}\right\} = 2q\ln\frac{d(u)}{d(1)} + \frac{g(u)}{\varepsilon}$.

By solving this equation we find the halfwidth of the contact region $d(u) = d(1)\sqrt{u}\exp[-g(u)/(2q\varepsilon)]$. In the particular case $g(u) = 0$ $d(u) = d(1)\sqrt{u}$ (generally speaking the contact is parabolic in shape with a curvilinear "axis").

It is noted that Eq. (4.7) has a class of accurate solutions with $q \equiv 0$ on condition that $g(u, v)$ with fixed u is in the form of a sum of an uneven function for the argument $2(v - v^0)/(v^+ - v^-)$ and a linear combination of Chebyshev polynomials (excluding the zero-order polynomial) from the same argument. It follows from this that the integral in the right-hand part of (4.7) equals zero under these conditions.

The procedure for obtaining (4.7) is also directly carried over to the case of Eq. (3.2), which may be written in the form

$$\int_{u^-}^{u^+} \frac{q(\xi) - q(u)}{2\left|\sin\frac{\xi - u}{2}\right|} d\xi + q(u)\ln\left\{\frac{1024\left|\operatorname{tg}\frac{u^+ - u}{4}\operatorname{tg}\frac{u - u^-}{4}\right|}{(v^+(u) - v^-(u))^2\varepsilon^2}\right\} = \frac{1}{\pi\varepsilon}\int_{v^-(u)}^{v^+(u)} \frac{g(u, v)\,dv}{\sqrt{(v^+(u) - v)(v - v^-(u))}}.$$

LITERATURE CITED

1. L. A. Galin, Contact Problems of Elasticity and Viscoelasticity Theory [in Russian], Nauka, Moscow (1980).
2. F. D. Gakhov, Boundary Problems [in Russian], Nauka, Moscow (1977).
3. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, Integrals and Series. Elementary Functions [in Russian], Nauka, Moscow (1981).
4. M. P. Kovalev and M. Z. Narodetskii, Design of Highly Accurate Ball Bearings [in Russian], Mashinostroenie, Moscow (1980).